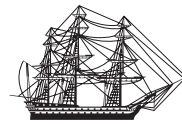


# Financial Simulation and Investment Expectations

Vanguard Investment Counseling & Research



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## Executive summary

Future investment returns are uncertain; no reliable means to foresee them exists. However, institutions and individuals making financial decisions need an approach to model the range of possible future returns. Any approach requires two fundamental ingredients:

- First, long-term historical returns must be examined to determine the attributes and interrelationships that characterize investment returns.
- Second, a simulation method is required to translate the observed attributes and interrelationships among investment returns into the potential range of future returns.

The variability or volatility around the average scenario in financial simulations often characterizes the risk of the investment decision under consideration.

Financial planners employ one of two statistical procedures to generate estimates of future asset values: historical simulation and Monte Carlo simulation. This paper appraises the relative merits and limitations of these approaches.

Author

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## Simulation approaches

Historical-simulation methods (e.g., time-path analysis) generate return scenarios by assuming that past events repeat in some chronological fashion. The premise is that potential changes in asset returns will not range beyond the changes previously observed in those assets' values over a defined historical period. Monte Carlo simulation methods, on the other hand, randomly generate future returns by specifying the likelihood that an asset will take a certain future value. To derive such probabilities, Monte Carlo uses the mean and variance that characterize a historical sample to impose explicit distributional assumptions in the simulation process.

Broadly speaking, the historical and Monte Carlo approaches are similar in that both rely on data from the past to generate future scenarios. The fundamental difference is that Monte Carlo simulation samples an asset's returns from an imposed "bell-curved" *distribution*, rather than by replicating chronological segments of the actual data *series*. Consequently, these two alternatives can lead to different expectations of investment risk.

## Four conclusions

Through an application of statistical analysis, financial theory, and portfolio simulation, this paper arrives at four general conclusions.

- Financial simulations should be based on historical observations over *decades* rather than shorter periods. This is essential to minimize the tendency for recent investment performance to unduly influence long-range risk assessment and asset allocation decisions. As a rule, investors should not expect future long-term returns to be significantly higher or lower than long-term historical returns for various asset classes and subclasses.
- Among the three historical-simulation frameworks (bootstrap resampling, rolling time path, and looping time path), looping time-path analysis most effectively addresses the dual objectives of preserving the empirical relationships among portfolio assets and ensuring an appropriate uniform sampling frequency. The other historical-simulation methodologies only meet one of the two objectives.
- Monte Carlo simulators possess distinct advantages, but overall they do not offer a superior alternative to appropriately conducted historical simulations. Monte Carlo simulation is better-equipped than historical simulations to generate "extreme-tailed" results—representing events that did not occur in the sampled period, but could have. However, the distributional assumptions of the Monte Carlo methodology introduce a potential bias to the assessment of investment risk.
- A regression-based simulation model can address these deficiencies by incorporating an empirical, time-varying structure to basic Monte Carlo simulation. Realizing the potential advantages of a regression-based Monte Carlo simulation tool, industry practitioners have begun to employ certain aspects of such time-series techniques.

## Realistic expectations of future performance

### Decisions required to run a financial simulation

Financial simulation requires realistic assumptions about three critical inputs:

1. An assumed value for the future average return on each asset (i.e., *mean*).
2. An estimate of the variability of returns around the mean (i.e., *standard deviation*) as a measure of risk.
3. An assumption about how the portfolio assets will co-vary, or react to changes in each other, over time (i.e., *correlation*).

**Forming expectations from long-run historical data.** It is best to sample data from as long a period as possible (subject to considerations of relevance discussed later). The more extensive the historical sample, the better it will approximate the mean and standard deviation (or return and risk) of the “true” return-generating process. Even though future events will not precisely replicate the past, a rich set of historical data provides a framework for developing reasonable expectations. Furthermore, long-run historical returns avoid certain biases that often characterize investor expectations.<sup>1</sup>

Historical performance is relevant to investment expectations because there are fundamental long-term relationships between the economy and the financial markets. Investment returns on a broad composite of corporate stocks or government bonds inevitably reflect the aggregate expansion or contraction in the values of goods and services produced in the economy.

#### **Practical issues: How long is the “long run?”**

Although it makes sense to rely on historically based expectations in investment planning and financial simulation, several practical issues exist. One is how to

define “long run.” Should analyses always incorporate the full extent of financial market data? How does one deal with profound structural changes that may make past results less applicable to the future?

For the purpose of forming expectations about future investment returns, it is best to draw on at least 30 years’ worth of historical data.<sup>2</sup> Thirty years should provide enough data to generalize the short-term responses of financial markets under several economic cycles, which on average tend to last five or six years. In addition, given that the causes and effects of business cycles and bear markets differ, a lengthy sample period captures the responses of asset classes to varying macroeconomic risk factors, such as inflation or interest rate fluctuations.

#### **Caveats in consulting long-run historical data.**

History can, at times, be misleading. Some periods have been characterized by profound structural change and their financial data are not directly relevant for current projections. *In short, subjective judgment is needed to determine the historical period that most reasonably summarizes expected investment returns.*

**Caveat #1: Political or social upheaval.** Structural changes in how various macroeconomic or political risk factors affect an asset class may make it advisable to exclude some previous periods from financial-planning models. For example, German and Japanese financial data predating World War II may be irrelevant in assessing the likelihood of future investment scenarios in those nations, given the subsequent radical change in their political systems (see Figure 1 on page 4). More recently, institutional and technological changes have radically altered how the futures, commodities, and foreign-exchange markets operate.

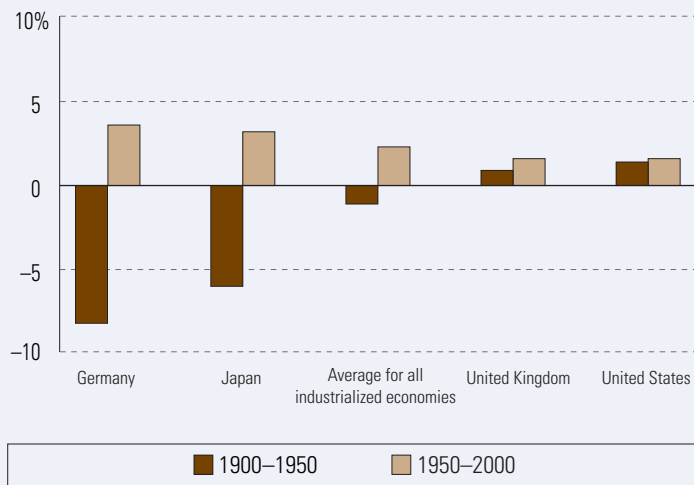
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1 Forward-looking expectations based upon investor surveys are often biased and may reflect desired returns, rather than expected returns. To be sure, changes in investor expectations can affect the relative return among assets, such as between stocks and bonds (i.e., the equity risk premium). However, research suggests that, in surveys, certain investors tend to extrapolate recent past performance for expected future returns. The significantly high correlation between recent investment performance and expected future performance in survey responses suggests that investors may form expectations adaptively and overweight the importance of the most recent events. This is one reason that high-quality historical investment data are often preferable to survey-based expectations.

2 Moreover, individuals planning for retirement often have investment horizons of at least 30 years; for certain institutional investors, this horizon extends indefinitely.

Figure 1. Drastic political or economic events can affect the relevance of financial data

Average annual inflation-adjusted returns of government bonds in four nations



Note: Germany returns exclude hyperinflation period of 1922-1923. The performance data shown represent past performance, which is not a guarantee of future results.  
Source: Dimson, Marsh, and Staunton (2000).

**Caveat #2: Poor or inappropriate data.** A lack of high-quality historical data, particularly for sub-asset classes, often forces investment professionals to rely only on more recent information. For analyses involving mutual funds or other primary asset-class benchmarks particularly fixed income indexes—reliable return data dates back only to the 1970s. Even when data appear copious, they may be inappropriate for use in financial simulations. For example, the use of appraisal data (rather than transaction statistics) in evaluating the return on private real estate investments results in artificially smooth returns and artificially low correlations with other asset classes, such as stocks or bonds. If taken at face value (as is often done in published academic studies), these dubiously low correlations could overstate the diversification benefits of investing in real estate holdings, thereby masking the true degree of risk in such an allocation.

## Quantifying investment uncertainty through simulation

### Historical-simulation methods

Historically based methods produce future portfolio scenarios by directly applying historical returns to a defined initial investment. The three historically based methods used in portfolio simulation are presented in Table 1.

For any given time series of investment returns, the only fundamental difference among the three approaches is their method of sampling the returns. To facilitate comparisons of the methods, this paper uses annual return data since 1960 when possible.

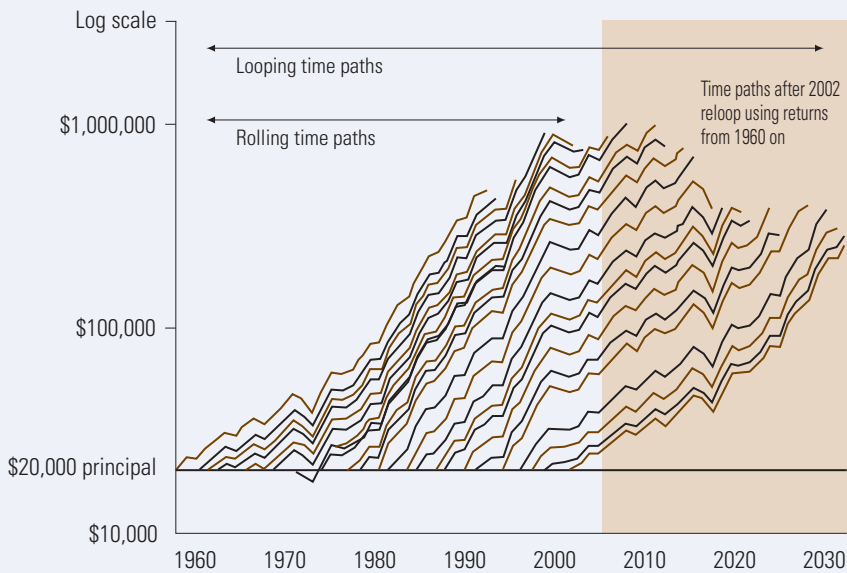
With this (or any) reference period, historical simulation incorporates three assumptions: First, that future mean returns will be about the same as the mean returns during the reference period. Second, that future investment conditions will be similar to the disparate investment and macro-economic environments experienced in the past. Third, and most important, that each future investment will follow a pattern similar to the actual pattern experienced over the historical period.

Table 1. Description of historical-simulation methods

Methodology	Description
Looping time-path analysis	Each financial simulation is generated by growing out the initial investment using chronological segments of historical asset-return data. When a particular scenario's investment horizon extends beyond the latest data point (i.e., 2003), the scenario returns to the beginning observation of the historical sample (i.e., 1960) and continues as above.
Rolling time-path analysis	Same as <i>looping time-path analysis</i> , but financial simulations terminate when the investment horizon reaches the end of the sample period.
Bootstrap resampling	Same as <i>looping time-path analysis</i> , except that the sequence of historical returns is randomly reordered.

Source: Vanguard Investment Counseling & Research.

Figure 2. An illustration of time-path analysis



Notes: Time paths are based on annual data from the years 1960–2002. Each path represents a \$20,000 portfolio bought and held for 30 years with no withdrawals. The portfolio is allocated 40% stocks, 40% bonds, and 20% cash, with returns based on the S&P 500 Index, the Ibbotson Corporate Bond Index, and 30-day U.S. Treasury bills, respectively. The data shown represent past performance, which is not a guarantee of future results.

### Time-path analysis

There are two varieties of time-path analysis: *rolling* and *looping*. Both methods create a set of sequential outcomes by looking at what would have happened to identical investments made at successive points in the past. They assume that a client begins investing at a specific date and then apply the actual returns for each subsequent period in generating a financial scenario, or time path. The use of rolling periods is popular among practitioners.

The time-path methodology is perhaps best explained through an example. Figure 2 presents simulated time paths for a three-asset portfolio held for 30 years. To create the initial path, the model first assumes that the client began investing in 1960; it then applies the actual asset returns for each subsequent year to the client's cash flow until the end of the investment horizon. These two steps yield a 30-year path for 1960–1989. To generate the next time path, the model pushes the initial investment date one year forward and repeats the process; and so on (e.g., Path #2: 1961–1990; Path #3: 1962–1991).

The only significant difference between *looping* and *rolling* time-path analyses is how each method handles the last observation in the historical data sample (i.e., 2002 in Figure 2). When the last actual observation is reached, the looping method returns to the beginning of the historical sample and reapplies the returns in an uninterrupted loop.

In Figure 2, this looping method creates post-2002 outcomes that follow distinct paths and have unique terminal values. The final number of paths generated in a looping simulation equals the number of observations in the historical sample period; for example, Figure 2 shows 43 time paths, each covering 30 years, based on annual return data from the 1960–2002 period. The portfolio risk in this hypothetical asset allocation can then be inferred from the wealth distribution of the 43 terminal time-path values.

The rolling time-path approach is simply a more restrictive version of the looping analysis. The rolling method counts only time paths with complete chronological sequences and so does not reuse historical data. As a result, it produces a smaller number of generated scenarios. In Figure 2, for example, the rolling method ends its 30-year samples with the 14th path, over the period 1973–2002.

### Bootstrap resampling

Both looping and rolling time paths generate *chronological* sequences from a series of historical returns. In contrast, the bootstrap method *randomly* reassembles returns from the same historical series. That is, for each holding period, bootstrapping randomly samples a set of asset returns until the number of drawn observations corresponds to the investment horizon.

For example, suppose that a planner wanted to use the 1960–2002 data to create bootstrap scenarios for a 30-year buy-and-hold portfolio. The first scenario might select data from 1973 for the

first year, following that with data from 1987, then data from 1964, and so forth until 30 years' worth of annual returns were assembled. Because the data can be reassembled in many ways, bootstrapping can produce a dramatically larger set of future scenarios than can time-path analysis. In addition, if the simulation permits years to be sampled more than once in a scenario (i.e., "bootstrapping with replacement"), the results can be used to derive a wider range of risk measurements from a given set of historical returns.

#### Advantages of looping time-path analysis

Table 2 lists the liabilities of each of the three historical-simulation methods.

**Looping advantage over rolling time paths: Uniform sampling frequency.** The looping method of time-path analysis has a distinct advantage over the more restrictive rolling method. Figure 3 on page 7, demonstrates the frequency with which each annual observation enters the 1960–2002 sample under all three historical-sampling techniques.

Because the rolling analysis may use only chronological sequences of historical data, it produces an undesirable "hill-topped" frequency distribution. The annual returns closer to the sample midpoint (i.e., 1970–1990 in Figure 3) are overweighted in the simulations because data from the years near the start and end points of the historical period "drop out" of the samples. This hill-topped sampling bias remains regardless of the length of the data series considered in a rolling time-path analysis (i.e., 5 years of data versus 100 years).

Both the looping method of time-path analysis and bootstrap resampling avoid this type of bias by sampling each observation in the historical period with the same frequency. The uniform (or flat-lined) sampling frequency in Figure 3 reflects the fact that these two methodologies do not emphasize some years more than other years in the simulated time paths.

**Looping time-path advantage over bootstrapping: Recognizing asset dependencies.** The identification of dependencies between financial assets is a key ingredient in financial simulation. Without question,

Table 2. Limitations of alternative historical simulation methods

Liability	Rolling time path	Looping time path	Bootstrap resampling
1. Future scenarios based on historical asset-return data	X	X	X
2. Future uncertainty expressed in terms of realized past events	X	X	X
3. Statistical problems in using overlapping multiperiod returns	X	X	
4. Frequency of historical sample endpoints underrepresented	X		
5. End-tailed events overweighted vis-à-vis more frequent events	•		
6. Destroys cross-sectional correlation among assets			
7. Destroys serial correlation for each asset class			X

X = Liability applies. • = Liability applies in certain cases.

Source: Vanguard Investment Counseling & Research.

the single greatest attribute of time-path analysis is that the modeling of actual historical returns preserves the empirical interrelationships among assets. These interrelationships or correlations may be *cross-sectional* or *serial*. Cross-sectional correlation represents the co-movement among assets at any given date. Serial (or time-based) correlation expresses how an asset's present return relates to its own past value, either positively (*persistence*) or negatively (*mean reversion*).

By design, time-path analysis preserves both of these correlations by generating intact paths directly from the historical data. Conversely, the bootstrapping method destroys any serial correlation by jumbling the order of the observations.

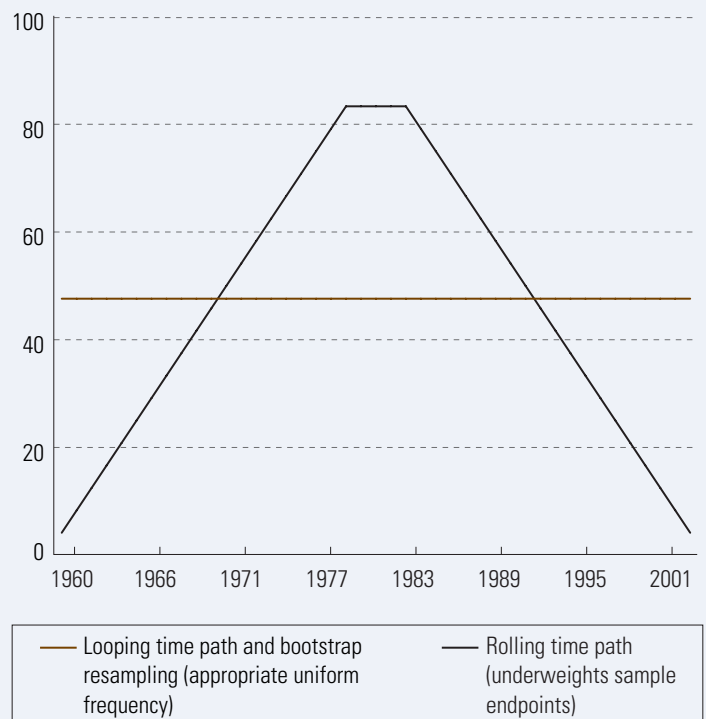
### Limitations of time-path analysis

Despite the strengths discussed, historical simulation does involve trade-offs. The chief problem is obvious: All of the scenarios generated by any historically based approach are determined by the same set of past returns. The time-path approach exacerbates this problem through its reliance on chronological sequences. As is clearly evident from Figure 2 on page 5, time-path analysis generates highly correlated asset scenarios that, as a group, do not provide any truly independent samples. In fact, for any contiguous pair of time paths, the only fully distinct information ("uncertainty") affecting the terminal values is the different start and end dates. Consequently, time-path analysis fundamentally understates investment risk.

The most serious limitation of time-path analysis is *truncation bias*, which is illustrated in Figure 4 on page 8. Since time-path analysis derives each path from the same historical series, the upper and lower bounds of any simulated asset return are strictly limited (or truncated) by whatever extremes exist in the historical data. Really terrible or terrific returns that *might* have occurred, but didn't, during the historical period are absent.

Figure 3. Historical sampling methods

Frequency distribution of annual sample observations since 1960

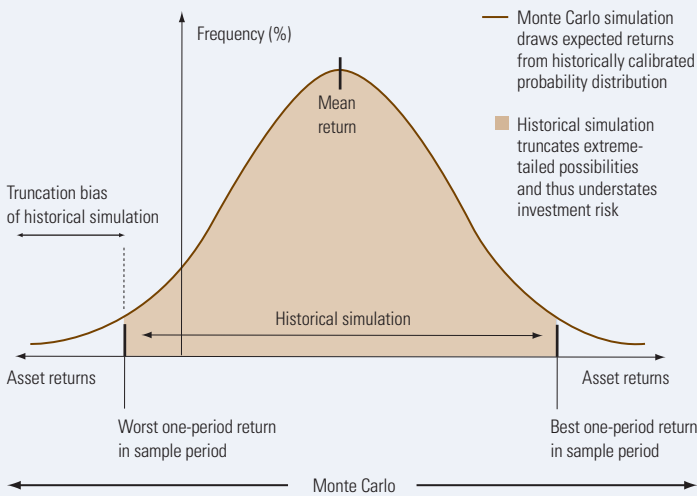


Source: Vanguard Investment Counseling & Research.

Truncation bias occurs because historically based simulations reflect merely the range of *observed* outcomes, rather than the range of *possible* outcomes. Historical simulation methods cannot produce scenarios for outlier events that do not appear within a given sample of investment returns. This fundamental bias associated with historical simulation exists even if higher-frequency monthly investment data are substituted for annual return data.<sup>3</sup>

<sup>3</sup> Substituting monthly return data for annual data naturally increases the number of simulated time paths for a given sample and thus more closely approximates the empirical distribution of past asset returns. For instance, monthly historical simulation over the 1960–2002 period generates 516 time paths rather than 43. However, altering the frequency of the time-path analysis does not eliminate the fundamental truncation bias because the 516 time paths remain a replication of historical events.

Figure 4. Limitations of historical simulation



Source: Vanguard Investment Counseling & Research.

Recent events offer a simple way to illustrate the danger posed by truncation bias. Using a hypothetical all-stock, buy-and-hold portfolio based on the S&P 500 Index, we can consider the results of excluding the latest bear market from a series of 20-year looping time paths. Figure 5 details the subsequent distribution of terminal portfolio wealth under various sample periods.

The omission of recent end-tailed events, namely the severe 2000–2002 bear market, has serious implications for simulated worst-case scenarios. The annualized expected portfolio return rises with each bear-market observation dropped from the sample, even though the best outcome (highest terminal wealth) is *identical* under every scenario. While the omission of extreme events in Figure 5 was intentional, it serves to illustrate that the best-case and worst-case scenarios derived from any historical simulation merely reflect the range of realized events.

### Simulation details of Monte Carlo analysis

The *multivariate normal distribution* serves as the benchmark asset-return model for Monte Carlo simulation, although alternative distributions can be used in financial modeling. The multivariate normal distribution assumes that the unconditional mean and standard deviation of asset returns are time-invariant, and that returns are independent from one year to the next. At each date over the simulation horizon, the random shocks generated by the multivariate normal model are adjusted so as to obey the average cross-sectional correlation observed in the historical data.

The correlation-based dependence structure for Monte Carlo analysis is derived through the Cholesky decomposition method. Technically, let  $L_1 = [l_{ij}]$  be the lower-triangular Cholesky decomposition of the correlation matrix  $C$  (that is,  $l_{ij} = 0$  for all  $j > i$  and  $C_1 = L_1 L_1^T$ ),  $\mu$  be the mean return on asset  $i$ , and  $\delta$  be the standard deviation of the return on asset  $i$ . Then, for a portfolio with  $n$  assets, the multivariate normal Monte Carlo model dictates that  $Z_i = \sum_{j=1}^n l_{ij} \xi_j$ , where  $\xi$  represents an independent standard normal random variable (i.i.d.  $\sim N(0,1)$ ) and where  $Z_i$  represents a correlated standard normal variable for asset  $i$ . The simulated return,  $r_i$ , on asset  $i$  then obtains as  $r_i = \mu_i + \delta_i Z_i$ .

## Monte Carlo simulation

**Methodology.** Monte Carlo simulation is among the most widely used scenario-generator tools in the financial industry. In one sense, Monte Carlo simulation is similar to historical simulation in that both rely on historical data to generate expected future asset paths. The fundamental difference is that Monte Carlo simulation abandons the deterministic assumption that past returns represent the full range of expected future returns. Specifically, the Monte Carlo approach samples an asset's returns from an imposed *probability distribution*, rather than by taking them chronologically from the historical data *series*.

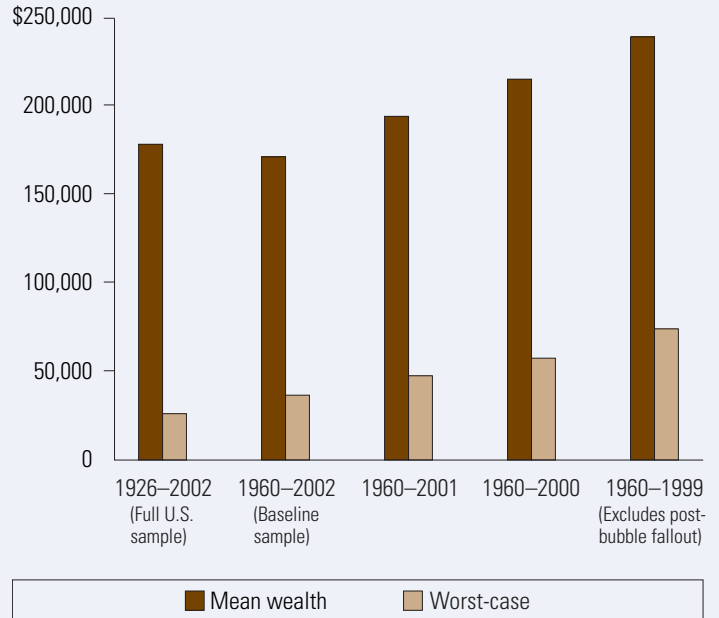
For each period within the financial simulation, Monte Carlo methods randomly draw asset returns from a probability distribution calibrated to conform to the historical properties of the asset. The calibrated probability distribution defines the likelihood that any given return will be drawn during the simulation process. Under the common assumption that asset returns are normally distributed like the bell curve in Figure 4 the mean, standard deviation, and covariance of each asset return are sufficient inputs to generate financial scenarios in the Monte Carlo procedure.

**Limitations.** Although Monte Carlo techniques do not suffer from truncation bias or certain other limitations of historical sampling, standard Monte Carlo methods have their own shortcomings. Table 3 on page 10, summarizes the relative drawbacks of the disparate approaches.

As commonly employed, Monte Carlo methods depart from a time-path-based approach by making three hallmark assumptions regarding asset returns in a portfolio. Specifically, standard Monte Carlo simulations assume an asset return to be:

1. Drawn from a multivariate normal (bell-shaped) distribution.
2. Uncorrelated with that asset's own past returns.
3. Fixed in its correlation with other asset returns in the portfolio.

Figure 5. Truncation of outlier events in historical simulation can understate downside risk



Note: Bars represent value of terminal portfolio wealth based on initial investment of \$20,000 held for 20 years.

Source: Vanguard Investment Counseling & Research.

Each of these assumptions is subject to criticism. Indeed, empirical research has documented numerous situations, involving both specific asset types and specific sample periods, where one or more of the three assumptions are invalid.

The most obvious shortcoming of the basic Monte Carlo approach is that the simulation algorithm imposes strict simplifying assumptions concerning the probability distribution of asset returns. In the same way that traditional mean-variance optimization tools calculate efficient portfolio frontiers, basic Monte Carlo simulation assumes that asset returns are distributed symmetrically around the mean return (i.e., in a bell-shaped probability distribution). However, for certain types of assets,

historical returns tend to deviate from the mean in ways not reflected by a bell curve. Such “fat-tailed” and “skewed” distributions raise serious questions about the validity of Monte Carlo simulations involving these types of assets, particularly with respect to high-frequency data (i.e., daily or weekly returns). And while Monte Carlo methods can incorporate other probability distributions in an attempt to better model the performance of given assets, these alternative assumptions are also subject to criticisms of measurement error, data mining, and sample dependency.

Likewise, the second assumption—that an asset’s returns are uncorrelated over time—can distort the simulation to a greater or lesser extent depending on the type of asset, the frequency of observations, and the historical period examined. Finally, empirical evidence often contradicts the third Monte Carlo assumption, that cross-asset correlation is fixed. For instance, the correlation between U.S. stocks and Treasury bonds has varied widely over time (see Figure 6). Historical simulation better captures these changes in the so-called equity risk premium.

#### Methodology comparison via portfolio simulation

To investigate how the limitations of Monte Carlo and looping time-path simulations affect the relative portrayal of investment risk, we can conduct a simulation experiment. For simplicity, the exercise considers a static asset-allocation strategy in which an investor deposits a hypothetical \$20,000 in a portfolio to be held for 30 years without withdrawals. The initial investment is allocated 40% to stocks (as represented by the S&P 500 Index), 40% to high-grade corporate bonds (Ibbotson data), and 20% to cash reserves (3-month Treasury bills). Each scenario generator uses annual benchmark data from 1960 through 2002.

In both cases, a simulation algorithm is employed to compound the returns generated each year from each asset, thus determining the level of wealth to be realized by the buy-and-hold investor at the end of each possible return scenario. Note that the average annual returns for stocks, bonds, and cash are equal for the historical and Monte Carlo simulations, but the way in which each simulation technique characterizes the uncertainty surrounding the average returns can lead to differences in the wealth values derived at the end of the 30-year period.

**Table 3. Limitations of historical and Monte Carlo simulation**

Liability	Looping time path	Basic Monte Carlo
1. Future scenarios adapted from historical asset returns	X (via replication)	X (via calibration)
2. Uncertainty limited to realized past events	X	
3. Number of unique scenarios constrained by number of observations in historical sample	X	
4. Assumes that financial returns are normally distributed		X
5. Destroys cross-sectional correlation among assets		•
6. Destroys serial correlation for each asset class		X

X = Liability applies.      • = Liability applies in certain cases.

Source: Vanguard Investment Counseling & Research.

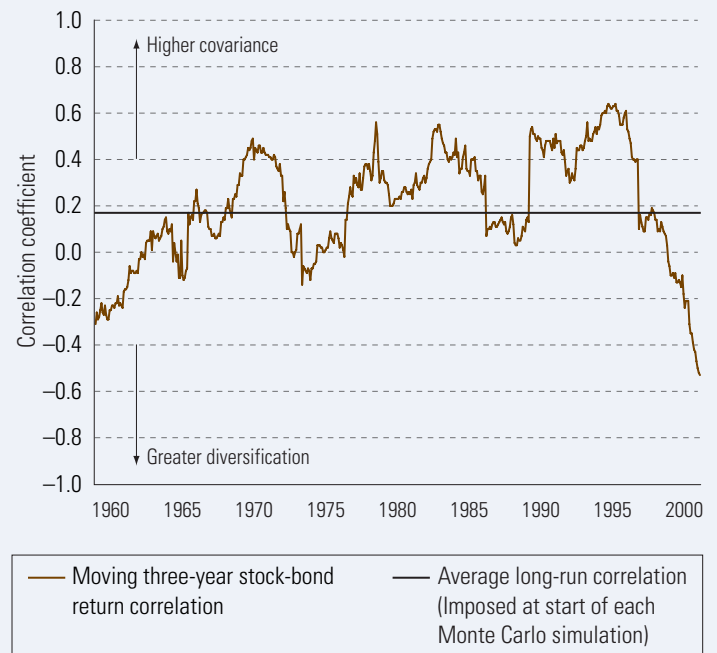
Table 4 on page 12 summarizes the features (i.e., mean and standard deviations) of the terminal wealth distributions for the 43 looping time paths and 1,000 Monte Carlo simulations, respectively. The table also provides an alternative risk measure known as Value-at-Risk, or VaR, which is useful to investors particularly concerned with downside risk.<sup>4</sup>

There are two salient features of Table 4. The first is the *similarity* in the average dollar values of the terminal portfolio values derived through each methodology. In fact, the median dollar values obtained from historical simulation and Monte Carlo simulation are nearly identical (~\$258,000). In this example, both techniques produce an average annual portfolio return of about 8.5%, an agreement that should be expected since the Monte Carlo algorithm draws from a distribution calibrated to the same total return series used in historical simulation.

The second critical feature in Table 4 is the marked difference in the statistical measures of variation—standard deviation and VaR—derived from the two simulation methodologies. The degree of financial risk appears significantly greater in the Monte Carlo simulation. This is a direct result of truncation bias in the looped time paths. Figure 7 on page 13 depicts the distribution of the terminal wealth values under the two simulation methods. The bars represent annualized returns, which are reported in percentiles. The figure places special emphasis on the “tails” of the distribution, namely those annualized returns in the top and bottom fifth percentiles.

These tail values differ notably under the two methodologies. The highest and lowest terminal values in the time-path simulation deviate less from the median return than the best-case and worst-case Monte Carlo simulations, reflecting the fact that historical simulation is unable to consider possible, albeit unlikely, events that could affect returns. Time-path analysis thus understates the risk (in terms of standard deviation) that is inherent in any distribution of historical asset returns. This truncation bias represents a fundamental shortcoming for *all* historical simulation methods, given that a primary

Figure 6. Stock-bond correlation and time-varying risk premium



Note: Stock returns calculated from the S&P 500 Index; bond returns based upon spliced Ibbotson/Lehman Intermediate Treasury Bond Index.  
Source: Vanguard Investment Counseling & Research.

necessity in financial planning is to envision the effect of rare downside events (i.e., bear markets). A Monte Carlo simulation better captures the end-tailed events. This is reflected in Table 4 by a standard deviation in average terminal wealth that is nearly *twice* that for time-path analysis.

The simulation results in Table 3 and Figure 7, however, do *not* imply that Monte Carlo simulation fully conveys the extent of financial risk. On the contrary, Monte Carlo distorts the risk calculation to some degree by failing to reflect the empirical interdependency among asset returns. The methodology does acknowledge a relationship between returns for different asset classes, but it does not allow that relationship to change; it is a constant at the start of the Monte Carlo algorithm.

<sup>4</sup> VaR quantifies the worst expected loss over the investment horizon under normal market conditions at a given confidence interval. The 99% VaR represents the terminal portfolio wealth value at the bottom 1% of the simulated distribution.

Consequently, the undeniable variation in the correlation between, say, stocks and bonds over time is not captured by the basic Monte Carlo approach. This restriction counteracts the otherwise noteworthy ability of Monte Carlo to simulate extreme events.

How the relative drawbacks of historical simulation and Monte Carlo simulation ultimately affect assessments of investment risk and potential future returns depends on the interaction of several factors. As a rule, the two approaches will be *closer* in their risk and return projections:

1. The shorter the investment horizon.
2. The longer the sample of the historical data.
3. The lower the variability in the comovement of asset returns over time (i.e., relatively stable cross-asset correlation).

4. The more closely a portfolio's historical asset returns resemble a normal distribution.

Points 1 and 2 minimize the impact of truncation bias in historical simulation. All else equal, a longer investment horizon widens the gap between the simulation approaches in the dispersion of terminal wealth because the effect of truncation bias grows over time. Points 3 and 4 minimize the shortcomings of the Monte Carlo approach vis-à-vis looping time-path analysis. In the simulation experiment, the empirical distributions of the annual returns of bonds and cash are more positively skewed and higher-peaked than the normal distribution assumed in Figure 4. Hence, the subsequent Monte Carlo simulation tends to understate above-average return realizations and to overstate dispersion relative to the historical attributes of bonds and cash. For

**Table 4. Alternative simulation results for a diversified portfolio**

	Deterministic (straight-line) method	Looping time path	Basic Monte Carlo
<b>Simulation assumptions</b>			
Cross-correlations among returns	None	Empirical	Estimated
Serial correlation in own returns	None	Empirical	None
Number of simulations	1	43	1,000
<b>Simulation results: Characteristics of terminal portfolio wealth</b>			
Mean terminal value (\$000)	310.5	267.2	309.0
Median terminal value (\$000)		258.1	257.5
Standard deviation (+ / - \$000)		115.4	211.6
99% Value-at-Risk (bottom 1% value, \$000)		127.3	76.6

This table contrasts the outcomes obtained by applying simulation techniques to a hypothetical \$20,000 portfolio bought and held for 30 years without liabilities or withdrawals.

The portfolio consists of 40% stocks, 40% high-grade corporate bonds, and 20% cash investments. The simulations are based on 1960–2002 annual returns for the S&P 500 Index, Ibbotson Corporate Bond Index, and 3-month Treasury bill.

**Notes:**

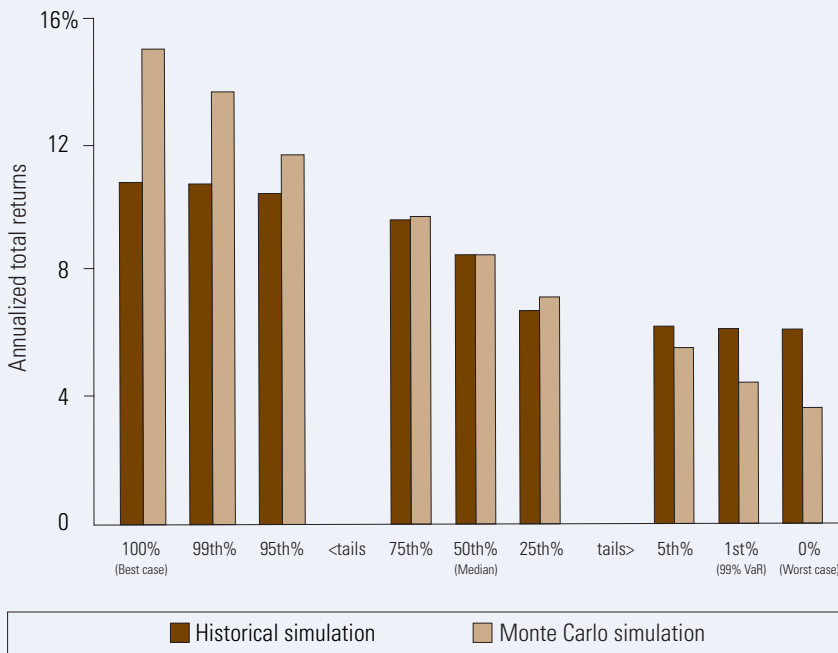
Monte Carlo results will vary slightly depending on the random-number seed chosen for simulation.

The deterministic method continuously compounds the initial investment for 30 years at the average return of each asset as defined over the 1960–2002 period (this is not genuine simulation because there is no return volatility).

Statistical tests fail to reject the null hypothesis that the median and mean terminal wealth values between simulation methods are equal. However, robust variance-equality tests (i.e., Levene/Brown-Forsythe tests) reject the null hypothesis that the standard deviations are equal; that is, one can state with more than 95% certainty that the dispersion in the terminal wealth distribution for the looping time-path analysis is significantly less than that generated by Monte Carlo.

Source: Vanguard Investment Counseling & Research.

Figure 7. Truncation bias of historical simulation



Notes: Bars represent annualized 30-year returns on a three-asset portfolio. The percentiles derive from underlying samples of 43 looped time paths and 1,000 Monte Carlo simulations, respectively.  
Source: Vanguard Investment Counseling & Research.

stocks, the opposite is true. Therefore, the overall effects in this case of imposing the multivariate normality assumption are generally offsetting, since the asset allocation is fairly evenly split between higher-variable stocks (40%) and less-variable fixed income securities (60%).

In sum, cases may arise in which—if cross-correlation between assets is low and the return distribution is far from bell-shaped—the Monte Carlo approach will lead to an overstatement of end-of-period mean wealth or a distortion in the future expected volatility.

### A regression-based approach to Monte Carlo simulation

To sum up the pros and cons of standard Monte Carlo: Although it captures low-probability events more effectively than historically based methods, the standard Monte Carlo technique largely destroys the interrelationships between asset returns that are explicitly captured in time-path analysis.

However, academics and practitioners have made significant progress in developing more robust simulation models based on the Monte Carlo approach. These newer, regression-based models:

- Preserve the time-based correlation in each asset's own returns.
- Capture the time-varying, cross-sectional correlation among returns from different assets.
- More effectively incorporate non-normality in the probability distribution of certain investment returns.
- Provide the capacity to incorporate macroeconomic risk factors.

Regression-based models incorporate many of these features by lending structure to basic Monte Carlo simulation. While a number of regression methods are in use, the most popular approach utilizes econometric time-series techniques known collectively as Vector Autoregression, or VAR (not to be confused with VaR, for Value-at-Risk). Since the early 1980s, VAR models have been widely used to make forecasts involving interrelated financial and economic time series and to analyze the impact of random disturbances on a system of variables.<sup>5</sup>

<sup>5</sup> Illustrating the relative strengths of this approach, some researchers have recently used regression-based simulation models to generate scenarios for applications in asset-liability management and retirement planning. See, for instance, Campbell et al. (2003) and Zenios and Zembia (2003). These developments extend actuarial models developed in the 1980s and 1990s to simulate future economic and investment conditions as “drivers” of expected asset returns (see Wilkie [1995]).

The concept of VAR estimation is simple: The expected return of each portfolio asset is estimated as a mathematical linear function of previous returns on *all* of the portfolio's assets as well as relevant macroeconomic predictive variables. An important distinction from other regression techniques is that the VAR model seeks to summarize only the linear *statistical* (rather than *structural*) relationship across asset classes and their risk factors over time. Consequently, VAR techniques should generate more-realistic scenarios precisely because the system of equations jointly models a portfolio of asset returns as both serially and cross-correlated processes. *In the unlikely event that all of the estimated parameters in the VAR model are zero, the regression-based approach reduces to basic Monte Carlo simulation.*

As in time-path analysis and Monte Carlo simulations, simulations using the VAR model require several assumptions, including:

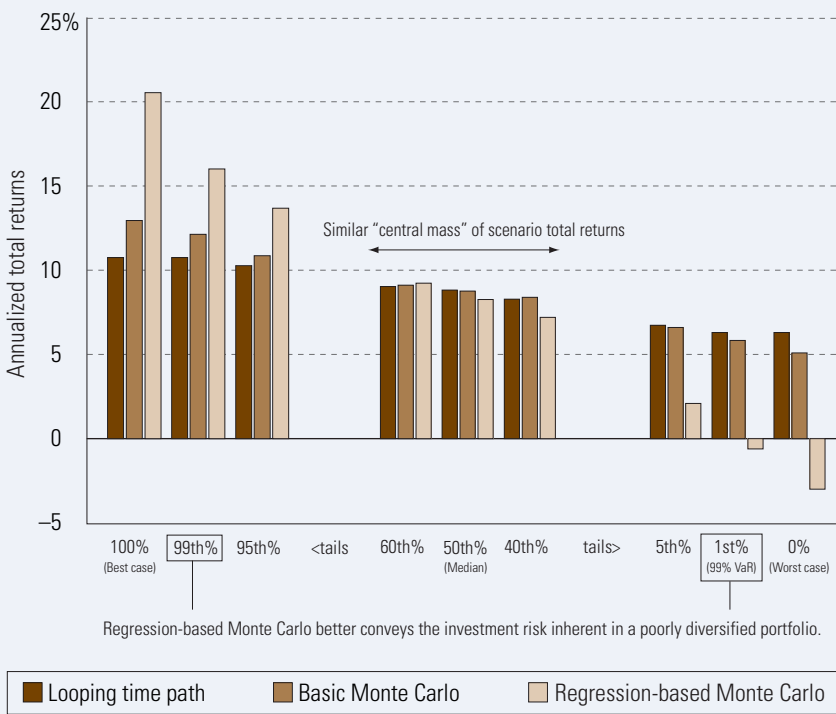
- The number of asset classes to include in the portfolio simulation.
- The frequency of the empirical and simulated returns (i.e., annual, monthly).
- The sample period of the historical data.

In the VAR-enhanced approach, the sample selection affects not only the calibration of the bell-shaped random draws (as in basic Monte Carlo) but also the empirical relationships estimated for the various assets (as realized in time-path analysis). A general rule should be to apply the model over the entire data sample since the VAR model (like all scenario generators) is sensitive to sample

selection. In addition, the user must choose the number of lagged terms in the VAR system to be estimated, which often can be determined through statistical tests of the appropriate lag length. For annual data, a first-order VAR is sufficient and unrestrictive, since a higher-order VAR can be rewritten mathematically as a one-year-lag VAR.

The potential benefits of the regression-based approach are perhaps best illustrated through a numerical exercise. Consider a buy-and-hold investor who allocates a hypothetical \$20,000 equally among three assets (in this case, three separate bond portfolios) that are to be held for 20 years without rebalancing or withdrawals. Furthermore, assume that the investor's portfolio is poorly diversified (an assumption that maximizes the differences among time-path analysis, basic Monte Carlo, and VAR-based Monte Carlo). Reflecting this poor diversification, the experiment specifies

Figure 8. Percentile comparisons of terminal portfolio watch



Source: Vanguard Investment Counseling & Research.

that the client has divided the money equally among indexed bond funds representing three highly correlated sectors: (1) long-term U.S. Treasuries; (2) intermediate U.S. Treasuries, and (3) long-term U.S. corporate bonds.

Figure 8 suggests the two main results of this exercise. The first is the similarity in the mean annualized returns. The second is that the VAR-enhanced Monte Carlo approach best brings out the potential for extreme returns in the poorly diversified portfolio.

The primary reason for more extreme end-tailed events in Figure 8 stems from the structure of the regression-based model. Specifically, the VAR simulation method appropriately transmits a severe random shock realized by one bond index fund to returns of the others, because all three bond funds would be affected in similar ways. Since the annual returns of the bond indexes are very highly correlated, systematic portfolio risk is not quickly “diversified away.” Thus the covariance of the shocks in the model is constant over time, but the correlation of the variables is time-varying precisely because past asset values are incorporated in the VAR model.

Realizing the potential advantages of a regression-based Monte Carlo simulation tool, industry practitioners have begun to employ certain aspects of such time-series techniques. Several firms (e.g., Financial Engines<sup>6</sup>) use variations of a regression-based approach to generate financial scenarios over long-run investment horizons. Potentially, the VAR-based simulation approach can be further enhanced to enrich the dimension of uncertainty in the data-generating process. As just one example, financial simulation could incorporate time-varying coefficients in the regression model. Academic research (Barberis, 2000) has demonstrated that portfolio simulation can be misleading if the allocation framework ignores the imprecision surrounding the parameter estimates of any regression model.

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<sup>6</sup> *Financial Engines* is a trademark of Financial Engines, Inc. Financial Engines Advisors LLC, a federally registered investment advisor and wholly owned subsidiary of Financial Engines, Inc., provides all advisory services.



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